Flat nearly Kähler manifolds

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Abstract

We classify flat strict nearly Kähler manifolds with (necessarily) indefinite metric. Any such manifold is locally the product of a flat pseudo-Kähler factor of maximal dimension and a strict flat nearly Kähler manifold of split signature $(2m, 2m)$ with $m \geq 3$. Moreover, the geometry of the second factor is encoded in a complex three-form $\zeta \in \Lambda^3(\mathbb{C}^m)^*$. The first nontrivial example occurs in dimension $4m = 12$.

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Contents

1 Introduction

Nearly Kähler geometry originated in the study of weak holonomy groups by Gray [G4]. In fact, nearly Kähler manifolds correspond to weak holonomy $U(n)$ and were intensively studied by Gray [G1, G2, G3]. These manifolds appear moreover in a natural way in the Gray-Hervella classification [GH] as one class of the sixteen classes of almost Hermitian manifolds.

Recent interest in nearly Kähler manifolds came from the fact, that in dimension 6 these manifolds are related to the existence of Killing spinors and that they admit a Hermitian connection with totally skew-symmetric torsion. Such connections were studied by Friedrich and Ivanov [FI] and are of interest in string theory. The classification of complete simply connected strict nearly Kähler manifolds was reduced to dimension 6 by Nagy [N1, N2]. Butruille [B] has shown that all strict homogenous nearly Kähler manifolds are 3-symmetric. These works are mainly concerned with Riemannian manifolds. In this paper we are especially interested in pseudo-Riemannian metrics.

The interest in flat nearly pseudo-Kähler manifolds is motivated by our study [S] of tt^*-structures (topological-antitopological fusion structures) on the tangent-bundle. In fact, flat nearly pseudo-Kähler manifolds provide an interesting class of tt^*-structures on the tangent-bundle. A second interesting class of solutions is given by *special Kähler manifolds* [CS]. In other words, one can interpret tt^*-structures on the tangent-bundle as a common generalisation of these two geometries. This duality can also be seen in this work. We construct flat nearly pseudo-Kähler manifolds of split signature from a certain constant three-form, while in [BC] flat special Kähler manifolds were constructed from a symmetric three-tensor.

Let us describe the structure and results of the paper. In the first section we recall some basic facts about flat nearly pseudo-Kähler manifolds (M, g, J). We give a self-contained presentation, with proofs which apply in the case of indefinite metrics and take advantage of the flatness of the metric. The essential points are the existence of a canonical connection ∇ with skew-symmetric torsion T and the properties of the tensor $\eta = \frac{1}{2}JDJ = D - \nabla = -\frac{1}{2}T$, where D is the Levi-Civita connection.

The classification results are then given in the second section. The first one is Theorem 1, which encodes a flat nearly pseudo-Kähler structure in a constant three-form η subject to two constraints. An explicit formula for J in terms of η is given. Next we analyze the constraints on η. It turns out that the first is equivalent to require that η has isotropic support (cf. Proposition 3) and the second is equivalent to a type condition on η (cf.

Theorem 2). We explicitly solve the two constraints on the real three-form η (in $4m$ variables) in terms of a freely specifiable complex three-form $\zeta \in \Lambda^3(\mathbb{C}^m)^*$. In particular, any such form $\zeta \neq 0$ defines a complete simply connected strict flat nearly pseudo-Kähler manifold, see Corollary 4.

Further we show that any strict flat nearly pseudo-Kähler manifold is locally the product of a flat pseudo-Kähler factor of maximal dimension and a strict flat nearly pseudo-Kähler manifold of dimension $4m \geq 12$ and split signature (Theorem 3). This implies, in particular, the non-existence of strict flat nearly Kähler manifolds with positive definite metric. The work finishes with the classification of complete simply connected flat nearly Kähler manifolds up to isomorphism in terms of $GL_m(\mathbb{C})$-orbits on $\Lambda^3(\mathbb{C}^m)^*$, see Corollary 5.

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2 Basic facts about flat nearly pseudo-Kähler manifolds

In this section we discuss some basic properties of nearly pseudo-Kähler manifolds. Since in this paper we are concerned with *indefinite* nearly Kähler manifolds with *flat* Levi-Civita connection, we give a self-contained discussion including indefinite metrics but specializing the general statements and proofs whenever possible using the flatness assumption. We have referred to the literature for more general statements in the positive definite case.

Definition 1 *An almost complex manifold* (M, J) *is called* almost pseudo-Hermitian *if it is endowed with a pseudo-Riemannian metric* g *which is* pseudo-Hermitian, *i.e. which satifies* $J^*g(\cdot,\cdot) = g(J\cdot, J\cdot) = g(\cdot,\cdot)$. *The nondegenerate two-form* $\omega := g(J\cdot,\cdot)$ *is called the* fundamental two-form.
An almost pseudo-Hermitian manifold (M, g, J) *is called* nearly pseudo-Kähler *manifold, if its Levi-Civita connection* D *satisfies the* nearly Kähler condition

$$(D_X J)Y = -(D_Y J)X, \quad \forall X, Y \in \Gamma(TM). \tag{2.1}$$

A nearly pseudo-Kähler manifold is called strict, *if* $DJ \neq 0$.

Proposition 1 *(cf. Friedrich and Ivanov [FI]) Let* (M, g, J) *be a nearly pseudo-Kähler manifold. Then there exists a unique connection* ∇ *with totally skew-symmetric torsion* T^∇ *satisfying* $\nabla g = 0$ *and* $\nabla J = 0$.
More precisely, it holds

$$T^\nabla = -2\eta \text{ with } \eta = \frac{1}{2} JDJ \tag{2.2}$$

and $\{\eta_X, J\} = 0$, *for all vector fields* X.

To prove the proposition we give two lemmas of independent interest.

Lemma 1 *Let (V, g, J) be a pseudo-Hermitian vector space and S a $(1,2)$ tensor, such that*

(i) S_X is skew-symmetric (with respect to g) for all $X \in V$,

(ii) $T(X,Y,Z) := g(S_X Y, Z) - g(S_Y X, Z)$, with $X, Y, Z \in V$, is totally skew-symmetric.

Then S, or more precisely $(X,Y,Z) \mapsto g(S_X Y, Z)$, is totally skew-symmetric, too.

Proof: It holds

$$\begin{aligned} g(S_X Y, Z) - g(S_Y X, Z) &\overset{(ii)}{=} -\left(g(S_Z Y, X) - g(S_Y Z, X)\right) \\ &\overset{(i)}{=} -g(S_Z Y, X) - g(S_Y X, Z), \end{aligned}$$

which implies $g(S_X Y, Z) = -g(S_Z Y, X)$. Together with property (i), this shows that S is totally skew-symmetric. □

Lemma 2 *Let (V, g, J) be a pseudo-Hermitian vector space and S a $(1,2)$ tensor satisfying:*

(i) S is totally skew-symmetric and

(ii) $[S_X, J] = 0$ for all $X \in V$.

Then S vanishes.

Proof: With arbitrary $X, Y, Z \in V$ we show

$$\begin{aligned} g(S_X Y, Z) &= g(JS_X Y, JZ) \overset{(ii)}{=} g(S_X JY, JZ) \\ &\overset{(i)}{=} -g(S_{JZ} JY, X) \overset{(ii)}{=} -g(JS_{JZ} Y, X) \\ &= g(S_{JZ} Y, JX) \overset{(i)}{=} -g(S_Y JZ, JX) \\ &\overset{(ii)}{=} -g(S_Y Z, X) \overset{(i)}{=} -g(S_X Y, Z). \end{aligned}$$

This shows $S = 0$. □

Proof: (of Proposition 1) First we show the uniqueness:
Let ∇ and ∇' be two such connections and $S := \nabla - \nabla'$ their difference tensor.
Then from $\nabla J = \nabla' J = 0$ we obtain $[S_X, J] = 0$ and from $\nabla g = \nabla' g = 0$ we get the skew-symmetry of S_X (with respect to g) for all vector fields X.
In addition

$$(T^{\nabla} - T^{\nabla'})(X,Y,Z) = g(S_X Y - S_Y X, Z)$$

is the difference of two totally skew-symmetric tensors and hence totally skew-symmetric. Lemma 1 implies that the tensor S is totally skew-symmetric and Lemma 2 shows the

uniqueness, i.e. the vanishing of S.
To prove the existence we define

$$\nabla := D - \eta \text{ with } \eta = \frac{1}{2} J D J.$$

The skew-symmetry of J yields that $D_X J$ is skew-symmetric (with respect to g). Further we have $\{J, D_X J\} = 0$, as follows from deriving $J^2 = -\mathbb{1}$, which shows that $\{\eta_X, J\} = 0$ and that $\eta_X = J D_X J$ is skew-symmetric for all vector fields X.
From the skew-symmetry of η_X and $Dg = 0$ we obtain $\nabla g = 0$.
Further we compute

$$\begin{aligned}\nabla_X J &= D_X J - \frac{1}{2}[J D_X J, J] = D_X J - \frac{1}{2}(J D_X J J - J^2 D_X J) \\ &= D_X J + J^2 D_X J = 0 \quad \forall\, X.\end{aligned}$$

This means $\nabla J = 0$. Finally we calculate the torsion

$$\begin{aligned}T^\nabla(X,Y) &= D_X Y - D_Y X - \eta_X Y + \eta_Y X - [X,Y] \\ &= T^D(X,Y) - (\eta_X Y - \eta_Y X) = -\eta_X Y + \eta_Y X \\ &= -2\eta_X Y,\end{aligned}$$

since DJ and consequently η is skew-symmetric by the nearly Kähler condition.
Since η_X is skew-symmetric for all X and $\eta_X Y = -\eta_Y X$ for all X, Y, η is totally skew-symmetric and $T^\nabla = -2\eta$ is totally skew-symmetric, too. □

Proposition 2 *Let (M, g, J) be a flat nearly pseudo-Kähler manifold. Then*

1) $\eta_X \circ \eta_Y = 0$ *for all* X, Y,

2) $D\eta = \nabla\eta = 0$.

Lemma 3 *(cf. Gray [G1]) Let (M, g, J) be a nearly pseudo-Kähler manifold. Then for all $X, Y \in TM$ it is*

$$g(R^D(X,Y)JX, JY) - g(R^D(X,Y)X, Y) = g((D_X J)Y, (D_X J)Y). \tag{2.3}$$

Proof: Since in this paper we are mainly concerned with flat nearly Kähler manifolds, we give a short proof under the additional assumption $R^D = 0$. With D-parallel vector fields X, Y we compute

$$\begin{aligned}0 &= g(R^D(X,Y)JX, JY) = g(D_X(D_Y J)X, JY) - g(D_Y \underbrace{(D_X J)X}_{=0}, JY) \\ &= X g((D_Y J)X, JY) - g((D_Y J)X, (D_X J)Y) \\ &= X[Y \underbrace{g(JX, JY)}_{const.} - g(JX, \underbrace{(D_Y J)Y}_{=0})] + g((D_X J)Y, (D_X J)Y) \\ &= g((D_X J)Y, (D_X J)Y).\end{aligned}$$
□

We linearize the identity $g((D_X J)Y, (D_X J)Y) = 0$ in Y to obtain

$$g((D_X J)Y, (D_X J)Z) + g((D_X J)Z, (D_X J)Y) = 0, \quad \forall X, Y, Z. \tag{2.4}$$

Lemma 4 *(cf. Gray [G2]) Let (M, g, J) be a flat nearly pseudo-Kähler manifold. Then*

$$g((D_X J)Y, (D_Z J)W) = 0 \quad \forall X, Y, Z, W. \tag{2.5}$$

Proof: Define the tensor $A(X, Y, Z, W) := g((D_X J)Y, (D_Z J)W) = A(Z, W, X, Y)$.
We know $A(X, Y, X, Z) = A(X, Z, X, Y) \overset{(2.4)}{=} -A(X, Y, X, Z)$ for all X, Y, Z, which implies $A(X, Y, X, Z) = 0$ and $A(Y, X, Z, X) = 0$.
We summarize the symmetries of A :

$$\begin{aligned} A(X, Y, Z, W) &= -A(Y, X, Z, W) = -A(X, Y, W, Z), \text{ (nearly Kähler condition)} \\ A(X, Y, Z, W) &= -A(Z, Y, X, W), \\ A(W, Y, Z, X) &= -A(W, Y, X, Z) = A(X, Y, W, Z) = -A(X, Y, Z, W), \end{aligned}$$

i.e. A is totally skew-symmetric.
In addition it holds

$$\begin{aligned} A(X, JY, Z, JW) &= g((D_X J)JY, (D_Z J)JW) \\ &= g(J(D_X J)Y, J(D_Z J)W) = A(X, Y, Z, W), \qquad (2.6) \\ A(X, Y, JZ, JW) &= g((D_X J)Y, (D_{JZ} J)JW) = -g((D_X J)Y, (D_W J)J^2 Z) \\ &= -g((D_X J)Y, (D_Z J)W) = -A(X, Y, Z, W). \qquad (2.7) \end{aligned}$$

The skew-symmetry of A yields

$$\begin{aligned} A(X, Y, Z, W) + A(X, Z, Y, W) &= 0, \\ A(X, JY, Z, JW) + A(X, Z, JY, JW) &= 0. \end{aligned}$$

The addition of these two equations gives

$$\begin{aligned} 0 &= A(X, Y, Z, W) + A(X, JY, Z, JW) + A(X, Z, Y, W) + A(X, Z, JY, JW) \\ &\overset{(2.6),(2.7)}{=} 2A(X, Y, Z, W) = 2g((D_X J)Y, (D_Z J)W) \end{aligned}$$

and the lemma is proven. □

Proof: (of Proposition 2)
1) From the last lemma we have

$$\begin{aligned} 0 = g((D_X J)Y, (D_Z J)W) &= -g((D_Z J)(D_X J)Y, W) \\ &= -g(J(D_Z J)J(D_X J)Y, W) \\ &= -4g(\eta_Z \, \eta_X Y, W). \end{aligned}$$

This shows $\eta_X \, \eta_Y = 0$ for all X, Y and finishes the proof of part 1).
2) With two vector fields X, Y we calculate

$$\begin{aligned} (D_X \eta)_Y &= D_X(\eta_Y) - \eta_{D_X Y} \\ &\overset{D=\nabla+\eta}{=} \nabla_X(\eta_Y) + [\eta_X, \eta_Y] - \eta_{D_X Y} \\ &= (\nabla_X \eta)_Y + \eta_{[\nabla_X Y - D_X Y]} + [\eta_X, \eta_Y] \\ &= (\nabla_X \eta)_Y - \eta_{\eta_X Y} + [\eta_X, \eta_Y] \\ &\overset{n.K.}{=} (\nabla_X \eta)_Y + \eta \eta_X Y + [\eta_X, \eta_Y] \overset{1)}{=} (\nabla_X \eta)_Y. \end{aligned}$$

Using $\eta = \frac{1}{2} JDJ$ and $\nabla J = 0$ we obtain

$$\nabla \eta = \frac{1}{2} J \nabla (DJ).$$

Therefore it is sufficient to show $\nabla(DJ) = 0$. We calculate for D-parallel vector fields:

$$\begin{aligned}
g(\nabla_X (DJ)_Y Z, W) &\overset{\nabla = D - \eta}{=} g(D_X (DJ)_Y Z, W) - g([\eta_X, D_Y J] Z, W) \\
&\overset{(*)}{=} g(D_X (DJ)_Y Z, W) \\
&\overset{DW=0}{=} X g((DJ)_Y Z, W) \\
&= X\,[(D_Y ø)(Z, W)] \\
&= D_X\,[(D_Y ø)(Z, W)] \\
&\overset{DY=DZ=DW=0}{=} (D^2_{X,Y} ø)(Z, W),
\end{aligned}$$

where $ø = g(J\cdot, \cdot)$. The second term in $(*)$ vanishes by part 1), since by $\{\eta_X, J\} = 0$ we get

$$J[\eta_X, D_Y J] = -\{\eta_X, J D_Y J\} = -2\{\eta_X, \eta_Y\} = 0.$$

The next lemma finishes the proof. $\square$

Lemma 5 *(compare Gray [G3] for the non-flat case)*
Let (M, g, J) be a flat nearly pseudo-Kähler manifold, then $D^2 ø = 0$.

Proof: The nearly Kähler condition is equivalent to

$$(D_X ø)(X, Y) = g((D_X J) X, Y) = 0, \ \forall X, Y. \tag{2.8}$$

Further $R^D = 0$ implies the symmetry of $D^2_{X,Y} ø$. Hence it suffices to show

$$(D^2_{X,X} ø)(Y, Z) = 0, \ \forall X, Y, Z.$$

Suppose X, Y, Z to be D-parallel. Then it is

$$\begin{aligned}
(D^2_{X,X} ø)(Y, Z) &= D_X\,[(D_X ø)(Y, Z)] \\
&\overset{(2.8)}{=} -D_X\,[(D_Y ø)(X, Z)] \\
&\overset{R^D=0}{=} -D_Y\,[(D_X ø)(X, Z)] \overset{(2.8)}{=} 0.
\end{aligned}$$

This yields the lemma. $\square$

3 Classification results for flat nearly pseudo-Kähler manifolds

We denote by $\mathbb{C}^{k,l}$ the complex vector space $(\mathbb{C}^n, J_{can})$, $n = k + l$, endowed with the standard J_{can}-invariant pseudo-Euclidian scalar product g_{can} of signature $(2k, 2l)$.

Let (M, g, J) be a flat nearly pseudo-Kähler manifold. Then there exists for each point $p \in M$ an open set $U_p \subset M$ containing the point p, a connected open set U_0 of $\mathbb{C}^{k,l}$ containing the origin $0 \in \mathbb{C}^{k,l}$ and an isometry

$$\Phi \,:\, (U_p, g) \tilde{\to} (U_0, g_{can}),$$

such that at the point p we have:

$$\Phi_* J_p = J_{can} \Phi_*.$$

In other words, we can suppose, that locally M is a connected open subset of $\mathbb{C}^{k,l}$ containing the origin 0 and that $g = g_{can}$ and $J_0 = J_{can}$.
From Proposition 1 and 2 we obtain:

Corollary 1 *Let $M \subset \mathbb{C}^{k,l}$ be an open neighborhood of the origin endowed with a nearly pseudo-Kähler structure (g, J) such that $g = g_{can}$ and $J_0 = J_{can}$. Then the $(1,2)$-tensor*

$$\eta := \frac{1}{2} J D J$$

defines a constant three-form on $M \subset \mathbb{C}^{k,l} = \mathbb{R}^{2k,2l}$ defined by

$$\eta(X, Y, Z) := g(\eta_X Y, Z)$$

satisfying

(i) $\eta_X \, \eta_Y = 0, \quad \forall X, Y,$

(ii) $\{\eta_X, J_{can}\} = 0, \quad \forall X.$

Conversely, we have the

Lemma 6 *Let η be a constant three-form on an open connected neighborhood $M \subset \mathbb{C}^{k,l}$ of 0 satisfying (i) and (ii) of Corollary 1. Then there exists a unique almost complex structure J on M such that*

a) $J_0 = J_{can}$,

b) $\{\eta_X, J\} = 0, \quad \forall X,$

c) $DJ = -2J\eta,$

where D stands for the Levi-Civita connection of the pseudo-Euclidian vector space $\mathbb{C}^{k,l}$. With $\nabla := D - \eta$ and assuming b), the last equation is equivalent to

c') $\nabla J = 0.$

Proof: The equivalence of c) and c') follows from a straightforward calculation. First we show the uniqueness of J:
Given two almost complex structures J and J' satisfying a)-c) we find

- $J_0 = J'_0$ and
- $\nabla J = \nabla J' = 0$,

which shows $J = J'$.
To show the existence we define

$$\begin{aligned} J &= \exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) J_{can} && (3.1)\\ &\overset{(i)}{=} \left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) J_{can}, && (3.2)\end{aligned}$$

where x^i are linear coordinates of $\mathbb{C}^{k,l} = \mathbb{R}^{2k,2l} = \mathbb{R}^{2n}$ and $\partial_i = \frac{\partial}{\partial x^i}$.
Claim: J is an almost complex structure satisfying a)-c).
a) From $x^i(0) = 0$ we obtain $J_0 = J_{can}$.
b) follows from equation (3.2) using (i) and (ii).
c) One computes

$$\begin{aligned} D_{\partial_j} J &= 2\exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\eta_{\partial_j} J_{can}\\ &\overset{(ii)}{=} \underbrace{-2\exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) J_{can}}_{J}\,\eta_{\partial_j} = -2J\,\eta_{\partial_j}.\end{aligned}$$

It remains to prove $J^2 = -Id$.

$$\begin{aligned} J^2 &= \left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) J_{can} \left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) J_{can}\\ &\overset{(ii)}{=} -\left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\left(Id - 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\\ &= -\left[Id - 4\left(\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)^2\right] \overset{(i)}{=} -Id.\end{aligned}$$

This finishes the proof. □

Theorem 1 *Let η be a constant three-form on a connected open set $U \subset \mathbb{C}^{k,l}$ containing 0 which satisfies (i) and (ii) of Corollary 1. Then there exists a unique almost complex structure*

$$J = \exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) J_{can} \tag{3.3}$$

on U such that

a) $J_0 = J_{can}$,

b) $M(U,\eta) := (U, g = g_{can}, J)$ *is a flat nearly pseudo-Kähler manifold.*

Any flat nearly pseudo-Kähler manifold is locally isomorphic to a flat nearly pseudo-Kähler manifold of the form $M(U,\eta)$.

Proof: (M,g) is a flat pseudo-Riemannian manifold. Due to Lemma 6, J is an almost complex structure on M and $J_0 = J_{can}$.
In addition it holds

$$J \quad = \quad J_{can} + \left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right) J_{can},$$

where $\{\eta_{\partial_i}, J\} = 0$ and η_{∂_i} is g-skew-symmetric. This implies that J is g-skew-symmetric. Finally from Lemma 6 c) and the skew-symmetry of η it follows the skew-symmetry of DJ. Therefore (M,g,J) is nearly pseudo-Kähler.
The remaining statement follows from Corollary 1 and Lemma 6. □

Now we discuss the general form of solutions of (i) and (ii) of Corollary 1. In the following we shall freely identify the real vector space $V := \mathbb{C}^{k,l} = \mathbb{R}^{2k,2l} = \mathbb{R}^{2n}$ with its dual V^* by means of the pseudo-Euclidian scalar product $g = g_{can}$.

Proposition 3 *A three-form* $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ *satisfies (i) of Corollary 1 if and only if there exists an isotropic subspace* $L \subset V$ *such that* $\eta \in \Lambda^3 L \subset \Lambda^3 V$*. If* η *satisfies (i) and (ii) of Corollary 1 then there exists a* J_{can}*-invariant isotropic subspace* $L \subset V$ *with* $\eta \in \Lambda^3 L$.

Remark: From the Proposition 3 we conclude that there are no strict flat nearly pseudo-Kähler manifolds of dimension less than 8. We shall see later that the dimension cannot be smaller than 12, see Corollary 2.

We define the support of $\eta \in \Lambda^3 V$ by

$$\Sigma_\eta := \mathrm{span}\{\eta_X Y \mid X, Y \in V\} \subset V. \tag{3.4}$$

Proof: (of Proposition 3) The proposition follows from the next two lemmas by taking $L = \Sigma_\eta$. □

Lemma 7 Σ_η *is isotropic if and only if* η *satisfies (i) of Corollary 1. If* η *satisfies (ii) of Corollary 1, then* Σ_η *is* J_{can}*-invariant.*

Proof: First the isotropy of Σ_η is equivalent to $g(\eta_X Y, \eta_Z W) = 0$ for all $X, Y, Z, W \in V$.
Further it holds

$$g(\eta_X Y, \eta_Z W) \overset{(*)}{=} -g(\eta_Z \eta_X Y, W). \tag{3.5}$$

In $(*)$ we used

$$g(\eta_X Y, Z) = \eta(X,Y,Z) = -\eta(X,Z,Y) = -g(\eta_X Z, Y) = -g(Y, \eta_X Z),\ \forall X, Y, Z.$$

Equation (3.5) shows $\eta_X \eta_Y = 0$ for all $X, Y \in V$ if and only if Σ_η is isotropic. The last assertion follows from

$$J\Sigma_\eta = \mathrm{span}\{J\eta_X Y \mid X, Y \in V\} \overset{(ii)}{=} \mathrm{span}\{-\eta_X JY \mid X, Y \in V\} = \Sigma_\eta.$$

$\square$

Lemma 8 *Let $\eta \in \Lambda^3 V$. Then $\eta \in \Lambda^3 \Sigma_\eta$.*

Proof: We take a complement $W \subset V$ of $L = \Sigma_\eta$. The decomposition

$$\Lambda^3 V = \bigoplus_{p+q=3} \Lambda^p L \wedge \Lambda^q W$$

induces a decomposition

$$\eta = \sum_{p+q=3} \eta^{p,q}.$$

Taking $X, Y \in L^\perp$ yields $L \ni \eta_X Y = \eta_X^{0,3} Y + \eta_X^{1,2} Y$. Now since $\eta_X^{0,3} Y \in W$ and $\eta_X^{1,2} Y \in L$, we get $\eta^{0,3} = 0$. Further the choice $X \in L^\perp$ and $Y \in W^\perp$ yields $\eta^{1,2} = 0$ and then the choice $X, Y \in W^\perp$ yields $\eta^{2,1} = 0$. This shows $\eta = \eta^{3,0}$. $\square$

Any three-form η on (V, J_{can}) decomposes with respect to the grading induced by the decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

into

$$\eta = \eta^+ + \eta^- \tag{3.6}$$

with

$$\eta^+ \in \Lambda^+ V := (\Lambda^{2,1} V + \Lambda^{1,2} V)^\rho$$

and

$$\eta^- \in \Lambda^- V := (\Lambda^{3,0} V + \Lambda^{0,3} V)^\rho,$$

where ρ is the canonical real structure on $V_{\mathbb{C}}$ with real-points V which extends to the exterior algebra.

Theorem 2 *A three-form $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ satisfies (i) and (ii) of Corollary 1 if and only if there exists an isotropic J_{can}-invariant subspace $L \subset V$ such that $\eta \in \Lambda^- L = (\Lambda^{3,0} L + \Lambda^{0,3} L)^\rho \subset \Lambda^3 L \subset \Lambda^3 V$. (The smallest such subspace L is Σ_η.)*

Proof: By Proposition 3, the conditions (i) and (ii) of Corollary 1 imply the existence of an isotropic J_{can}-invariant subspace $L \subset V$ such that $\eta \in \Lambda^3 L$. The next lemma shows that the condition (ii) is equivalent to $\eta \in \Lambda^- V$. Therefore $\eta \in \Lambda^3 L \cap \Lambda^- V = \Lambda^- L$. The converse statement follows from the same argument. $\square$

Lemma 9 *It is*

$$(\Lambda^{3,0}V+\Lambda^{0,3}V)^\rho = \{\eta \in \Lambda^3 V \mid \eta(\cdot, J\cdot, J\cdot) = -\eta(\cdot,\cdot,\cdot)\} = \{\eta \in \Lambda^3 V \mid \{\eta_X, J\} = 0,\ \forall X \in V\}.$$

Proof: We have the decomposition

$$\Lambda^2 V = (\Lambda^{1,1}V)^\rho \oplus (\Lambda^{2,0}V + \Lambda^{0,2}V)^\rho,$$

where

$$(\Lambda^{1,1}V)^\rho = \{\alpha \in \Lambda^2 V | \alpha(J\cdot, J\cdot) = \alpha\} \cong \{A \in \mathfrak{so}(V) | [A, J] = 0\}$$

and

$$(\Lambda^{2,0}V + \Lambda^{0,2}V)^\rho = \{\alpha \in \Lambda^2 V | \alpha(J\cdot, J\cdot) = -\alpha\} \cong \{A \in \mathfrak{so}(V) | \{A, J\} = 0\}.$$

This induces the following direct decomposition:

$$V \otimes \Lambda^2 V = V \otimes (\Lambda^{1,1}V)^\rho + V \otimes (\Lambda^{2,0}V + \Lambda^{0,2}V)^\rho.$$

We claim that

$$(V \otimes (\Lambda^{2,0}V + \Lambda^{0,2}V)^\rho) \cap \Lambda^3 V = (\Lambda^{3,0}V + \Lambda^{0,3}V)^\rho.$$

The claim implies the lemma. To see the claim, let us first observe the following obvious inclusion:

$$(V \otimes (\Lambda^{2,0}V + \Lambda^{0,2}V)^\rho) \cap \Lambda^3 V \supset (\Lambda^{3,0}V + \Lambda^{0,3}V)^\rho.$$

To show the equality we observe that an element of $V \otimes (\Lambda^{2,0}V + \Lambda^{0,2}V)^\rho$ is totally skew if and only if its four components in

$$V^{1,0} \otimes \Lambda^{2,0}V, \quad V^{1,0} \otimes \Lambda^{0,2}V, \quad V^{0,1} \otimes \Lambda^{2,0}V \quad \text{and} \quad V^{0,1} \otimes \Lambda^{0,2}V$$

are totally skew. To finish we notice that

$$(V^{1,0} \otimes \Lambda^{2,0}V + V^{0,1} \otimes \Lambda^{0,2}) \cap \Lambda^3 V = (\Lambda^{3,0}V + \Lambda^{0,3}V)^\rho$$

and

$$(V^{1,0} \otimes \Lambda^{0,2}V + V^{0,1} \otimes \Lambda^{2,0}) \cap \Lambda^3 V = 0.$$

□

Corollary 2 *There are no strict flat nearly pseudo-Kähler manifolds of dimension less than 12.*

Proof: By Theorem 1 and 2 any flat nearly pseudo-Kähler manifold M is locally of the form $M(U,\eta)$, where $\eta \in \Lambda^- L$ for an isotropic J_{can}-invariant subspace $L \subset V$ and $U \subset V$ is an open subset. $M(U,\eta)$ is strict if and only if $\eta \neq 0$, which is possible only for $\dim_{\mathbb{C}} L \geq 3$, i.e. for $\dim M \geq 12$. □

Theorem 3 *Any strict flat nearly pseudo-Kähler manifold is locally a pseudo-Riemannian product $M = M_0 \times M(U,\eta)$ of a flat pseudo-Kähler factor M_0 of maximal dimension and a strict flat nearly pseudo-Kähler manifold $M(U,\eta)$ of (real) signature $(2m, 2m)$, $4m = \dim M(U,\eta) \geq 12$. The J_{can}-invariant isotropic support Σ_η has complex dimension m.*

Corollary 3 *Let (M, g, J) be a flat nearly Kähler manifold with a (positive or negative) definite metric g then $\eta = 0$, $\nabla = D$ and $DJ = 0$, i.e. (M, g, J) is a Kähler manifold.*

Proof: (of Theorem 3) By Theorem 1 and 2, M is locally isomorphic to an open subset of a manifold of the form $M(V, \eta)$, where $\eta \in \Lambda^3 V$ has a J_{can}-invariant and isotropic support $L = \Sigma_\eta$. We choose a J_{can}-invariant isotropic subspace $L' \subset V$ such that $V' := L + L'$ is nondegenerate and $L \cap L' = 0$ and put $V_0 = (L + L')^\perp$. Then $\eta \in \Lambda^3 V' \subset \Lambda^3 V$ and $M(V, \eta) = M(V_0, 0) \times M(V', \eta)$. Notice that $M(V_0, 0)$ is simply the flat pseudo-Kähler manifold V_0 and that $M(V', \eta)$ is strict and of split signature $(2m, 2m)$, where $m = \dim_{\mathbb{C}} L \geq 3$. □

For the rest of this paper we consider the case $V \cong \mathbb{C}^{m,m}$ and denote a maximal J_{can}-invariant isotropic subspace by L. We will say that a complex three-form $\zeta \in \Lambda^3(\mathbb{C}^m)^*$ has *maximal support* if $\mathrm{span}\{\zeta(Z, W, \cdot) | Z, W \in \mathbb{C}^m\} = (\mathbb{C}^m)^*$.

Corollary 4 *Any non-zero complex three-form $\zeta \in \Lambda^{3,0} L \cong \Lambda^3(\mathbb{C}^m)^*$ defines a complete flat simply connected strict nearly pseudo-Kähler manifold $M(\eta) := M(V, \eta)$, $\eta = \zeta + \bar{\zeta} \in \Lambda^3 L \subset \Lambda^3 V$, of split signature. $M(\eta)$ has no pseudo-Kähler de Rham factor if and only if ζ has maximal support.*

Conversely, any complete flat simply connected nearly pseudo-Kähler manifold without pseudo-Kähler de Rham factor is of this form.

Proof: This follows from the previous results observing that the support of η is maximally isotropic if and only if ζ has maximal support. □

Corollary 5 *The map $\zeta \mapsto M(\zeta + \bar{\zeta})$ induces a bijective correspondence between $GL_m(\mathbb{C})$-orbits on the open subset $\Lambda^3_{reg}(\mathbb{C}^m)^* \subset \Lambda^3(\mathbb{C}^m)^*$ of three-forms ζ with maximal support and isomorphism classes of complete flat simply connected nearly pseudo-Kähler manifolds $M(\zeta + \bar{\zeta})$ of real dimension $4m \geq 12$ and without pseudo-Kähler de Rham factor.*

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